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Differential Bases of Emergy Algebra

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ABSTRACT

The well-known rules of Emergy Algebra, originally formulated in steady state conditions, are reconsidered and analyzed from a dynamic point of view. In such a sense the paper points out their corresponding differential bases. The latter, in turn, represent the preferential guide to recognize their most profound physical meaning.
However, for the sake of completeness, a possible generalization of the same rules from steady state to variable conditions is also considered.
The analysis is particularly focused on the three fundamental generative *processes represented by co-production, inter-action, and feed-back, which are formally described* (*under dynamic conditions*) *by means of the Incipient Fractional Differential Calculus. In so doing, the mathematical method adopted succeeds in defining the output exceeding Quality of the mentioned processes by means of the corresponding* Ordinality *of their associated output Transformities.*
Such a dynamic analysis enables us to show that the rules of Emergy Algebra proposed by Prof. Odum under steady state conditions have a well-founded dynamic physical nature, adequately described by the differential operators adopted. The analysis also shows that the originally conceived rules of Emergy Algebra continue to hold even when the dynamics of a process becomes extremely complicated.

INTRODUCTION

The basic rules of Emergy Algebra can be summarized as follows:
1. "*All Source Emergy to a Process is assigned to the Process's output*"
2. "*By-products from a Process have the total Emergy assigned to each pathway*"
3. "*When a pathway splits, the Emergy is assigned to each* "*leg*" *of the split based on their percent of the total Exergy flow on the pathway*"[1]
4. "*Emergy cannot be counted twice within a system. In particular*:
   a) by-products, when reunited cannot be summed;
   b) Emergy in feedbacks should not be double counted" (Brown 1993; Brown & Herendeen 1996).

For the sake of completeness, it is worth adding a fifth rule concerning a more sophisticated process termed as *Interaction*:
5. "*Output Emergy of an interaction Process is proportional to the product of the Emergy inputs*" (Odum, 1994a).

A rapid glance at the above-mentioned rules allows us to immediately point out that:

[1] The third rule directly refers to *Exergy* in accordance with the general definition of Emergy (Giannantoni 2000a, 2001c)

- The first rule represents a sort of "closure" rule in the case of one sole output;
- The second rule is extremely important because it shows how *co-generative* processes represent the basic processes mostly responsible for the increase in Emergy in self-organizing systems;
- The third rule points out the basic distinction between a *co-production* process and a simple *split* process (a mere subdivision of a flow into two equivalent sub-flows);
- Rule 4a) prevents erroneous accounting which would lead to an "artificial" amplification of Emergy, not related to a generative process (such as, for instance, a co-production).Thus it prevents double-counting of an identical contribution, already accounted for in its primary generative phase;
- Rule 4b) can be simply seen as a particular case explicitly pointed out for the sake of clarity.

The fifth rule pertains to *inter-action*. It completes the list of basic processes capable of generating an exceeding Emergy and also enables us to consider the feedback process as a particular form of self-interaction. In such a perspective the basic generative processes are: *co-production*, *inter-action* and *feed-back*.

It is also known that such rules are assumed as being valid under steady state conditions and are also used, without any modification or basic justification under stationary or slow transient conditions.

One fundamental problem is thus represented by the extension of their validity to variable conditions. In this case it is of primary importance to recognize whether they are well-founded in differential terms (which is equivalent to research for *their basic dynamic foundations*). This aspect shows all its relevance if we take into account that the rules of Emergy Algebra represent an *essential part* of the definition of Emergy. In fact Emergy is rigorously defined on the basis of two distinct elements: a correct dynamic Balance Equation (i.e. accounting rules) and an additional assumption concerning its reference level (i.e., solar Emergy with an associated conventional value of Transformity, for instance 1 seJ/J) (Giannantoni, 2000a).

We will thus analyze, in a rapid sequence, the three most important *generative* processes (*co-production*, *inter-action*, *feed-back*) and their associated dynamic foundation expressed in differential terms. For the sake of completeness, we will also mention split processes.

SHORT REMINDERS ABOUT MATHEMATICAL METHODS ADOPTED

The Rules of Emergy Algebra enabled Odum not only to show, but also to account for, an *extra-ordinary* aspect pertaining to living systems: their intrinsic capacity of generating ever new forms of processes, characterized by an *ever-increasing level of Quality*. This new concept of quality (thus indicated by a capital Q) is *not understood* as a simple property or a characteristic of a particular phenomenon or process, but it is recognized as being any *emerging* property that is *not reducible* to its phenomenological premises or to our traditional mental categories. This recent concept of science is inducing a profound revision in Classical Thermodynamics and in several related disciplines. It suggests that we modify our language to adequately describe the dynamics of Quality. In particular, this includes the *formal language* represented by mathematics, which is recognized as the most appropriate linguistic form adopted by science.

To this purpose a new form of derivative, the *incipient* derivative, represented by $\tilde{d}/\tilde{d}t$ has been introduced (Giannantoni, 2001d, 2002a) to better describe the dynamics of living systems. This derivative also allows processes to be modeled as *intrinsically linear* and always yields *explicit solutions*.

In order to include the initial conditions in the differential equation modeling the process a new *generator* $\tilde{D}$ was introduced

$$\tilde{D}\, f(t) = \frac{\tilde{d}}{\tilde{d}t}\, f(t) + f(0)\,\tilde{\delta}(t) \qquad (2.1)$$

where $\tilde{\delta}(t)$ represents the Dirac Delta function.
In such a way a differential equation of order n, with variable coefficients, takes the form

$$P_n(t, \frac{\tilde{d}}{\tilde{dt}}) f(t) = \tilde{g}(t) \quad (2.2)$$

where $$P_n(t, \frac{\tilde{d}}{\tilde{dt}}) = \frac{\tilde{d}^n}{\tilde{dt}^n} + a_{n-1}(t)\frac{\tilde{d}^n}{\tilde{dt}^n} + ... + a_1(t)\frac{\tilde{d}}{\tilde{dt}} + a_o(t) \quad (2.3).$$

Equation (2.2) can be simply written as

$$P_n(t, \tilde{D}) f(t) = (\sum_{k=0}^{n-1} c_k \tilde{D}^{n-1-k})\tilde{\delta}(t) + \tilde{g}(t) \quad (2.4)$$

where the coefficients c_k are related to the initial conditions

$$f^{(k)}(0) = f_{k0} \qquad (k = 1,2,....,n-1) \quad (2.5).$$

The obtained structure (2.4) is extremely important because, when re-written in the form

$$f(t) = P_n(t, \tilde{D})^{-1}[(\sum_{k=0}^{n-1} c_k \tilde{D}^{n-1-k})\tilde{\delta}(t) + \tilde{g}(t)] \quad (2.6)$$

it gives the explicit solution to Eq. (2.2), with the initial conditions (2.5). In this respect it is worth noting that the various functions of $\tilde{D}$, such as, for instance, $P_n(t, \tilde{D})$, play the analogous role that is played by Laplace Transform, while the *generator* $\tilde{D}$ plays the same role as the complex variable s.
However the fundamental differences between the two methods are: i) the method based on the generator $\tilde{D}$ always operates in the domain of the time variable t, and thus it always maintains an evident *physical meaning*, whereas the functions of complex variable s rarely have an explicitly interpretable physical meaning; ii) the method based on the generator $\tilde{D}$ is always applicable to both linear and "non-linear" differential equations (because the latter are always *intrinsically linear* when interpreted in terms of $\tilde{D}$), whereas the method based on Laplace Transform yields negligible, or even no advantages in the case of traditional *non-linear* differential equations.
The three fundamental rules of Emergy Algebra *co-production*, *inter-action* and *feed-back* will be analyzed in three successive sections by means of the above-mentioned Incipient Differential Calculus in order to give differential bases to Emergy Algebra. At the same time, it will show the intimate structural nature of the corresponding *Emergy Source Terms.* The latter are always present, with their specific contribution, in the mathematical formulation of the Maximum Em-Power Principle (Giannantoni 2001c,d, 2002a).

CO-PRODUCTION

The co-production process has already been analyzed in (Giannantoni, 2001d). However it will represented here in more detail.
The process can be schematized as in Fig. 3.1, where $\Phi(u)$ represents the Emergy Source Term. The pertinent Emergy Balance *in steady state conditions* (but often adopted also in stationary or slow transient conditions), can be thus written as follows

$$\dot{E}m(u) + \Phi(u) = \dot{E}m(y_1) + \dot{E}m(y_2) \tag{3.1}$$

where

$$\dot{E}m(y_1) = \dot{E}m(y_2) \quad (3.2) \qquad \text{and} \qquad \Phi(u) = \dot{E}m(u) \quad (3.3).$$

It is easy to recognize that condition (3.2), corresponding to the second Rule of Emergy Algebra, is assumed as being valid only on the basis of the *relational structure* of the outputs of the co-production process, that is *without considering* its internal *productive structure*, which, in any case, does not appear explicitly in Eq. (3.1), apart from the explicit assumption of the *uniqueness* of the same process. In other words the latter is analyzed in terms of its mere *phenomenological* characteristics, as if it were a *black box*.

We want now to show the basic reason for condition (3.2), not only in steady state conditions but also in variable conditions. In this latter case we have to take the Emergy Accumulation Term into account, so that Eq. (3.1) becomes (see Giannantoni, 2001d)

$$\dot{E}m(u) + \Phi(u) = \frac{\tilde{\partial}}{\tilde{\partial} t} A_D + \dot{E}m(y_1) + \dot{E}m(y_2) \tag{3.4}$$

where the "dot" notation for the derivative is understood as an incipient derivative like in the term $\tilde{\partial} A_D / \tilde{\partial} t$, which represents the "local" variation (in the Eulerian sense) of the Accumulated Emergy (A_D) on behalf of the considered system (geometrically defined by the domain D).

In order to reach a more adequate description of the internal productive structure, let us now compare Eq. (3.4) with a *fractional differential equation*, always in terms of incipient derivatives, written in a unique variable Em (already thought of as a *flow*, for simplicity of notation), whose homogeneous part is "similar" to Eq. (3.4), that is

$$C \cdot \frac{\tilde{\partial}}{\tilde{\partial} t} Em(t) + A \cdot \frac{\tilde{d}^{1/2}}{\tilde{d} t^{1/2}} Em(t) + B \cdot Em(t) = Em[u(t)] \quad (3.5),$$

where the symbol $\tilde{d}/\tilde{d}\, t$, in the most general case, represents the incipient Lagrangian derivative, that is

$$\frac{\tilde{d}^{1/2}}{\tilde{d} t^{1/2}} = \left(\frac{\tilde{\partial}}{\tilde{\partial} t} + v \cdot \tilde{\nabla} \right)^{1/2} \tag{3.6}$$

in which v is the velocity of the mass flow and $\tilde{\nabla}$ is the incipient *nabla* "generator" (understood as a *prior* operator).

It is then easy to show that, if we assume the output Emergy Flow to be proportional to the accumulated Emergy (as is usual in physical and biological systems)

$$Em = k \cdot A_D \tag{3.7}$$

it follows that, for $\quad C = k \quad A = 1 \quad$ and $\quad B = 1 \quad$ (3.8),

Eq. (3.5) represents the *most general dynamic model* of the considered process. In order to find its general solution, let us then reduce Eq. (3.5) to its standard form (as a linear differential equation of the first order with constant coefficients)

$$\frac{\tilde{d}}{\tilde{d}\,t} Em(t) + K_1 \cdot \frac{\tilde{d}^{\,1/2}}{\tilde{d}\,t^{1/2}} Em(t) + K_2 \cdot Em(t) = K_3 \cdot Em[u(t)] \tag{3.9}$$

where $K_1 = A/C \qquad K_2 = B/C \qquad K_3 = 1/C$ (3.10).

In fact, in adherence to the assumptions according to which Eq. (3.4) was formulated, we will limit our considerations to case of a simple Eulerian description. This is equivalent to assume that $v = 0$ or, alternatively, to deal with the problem in terms of the sole local time derivatives ($\tilde{\partial}/\tilde{\partial}\,t$) (that is in the absence of any spatial gradient). Such a partial time derivative, however, for the sake of simplicity will be always represented as usual ($d\,/\,\tilde{d}\,t$), without any possibility of confusion.

If we now search for solutions to the associated homogeneous equation through functions of the form $e^{\alpha^2 t}$ and we take into account that the fractional derivative of order ½ of the exponential function gives *two distinct values* such as (see Giannantoni 2001d)

$$\frac{d^{\,1/2}}{dt^{1/2}} e^{\alpha^2 t} = \pm\alpha \cdot e^{\alpha^2 t} \tag{3.11}$$

we obtain a solution in the form of a “binary” function, that is structured as follows

$$\tilde{E}\,m(y(t)) = \begin{pmatrix} \tilde{E}\,m(y_1(t)) \\ \tilde{E}\,m(y_2(t)) \end{pmatrix} = \begin{pmatrix} c_{11} \\ c_{22} \end{pmatrix} \cdot e^{\begin{pmatrix} \alpha_{11} \\ \alpha_{22} \end{pmatrix}^2 t} + \begin{pmatrix} c_{12} \\ c_{21} \end{pmatrix} \cdot e^{\begin{pmatrix} \alpha_{12} \\ \alpha_{21} \end{pmatrix}^2 t} \tag{3.12}$$

The exponents $\alpha_{1,i}$ and $\alpha_{2,i}$ are the solutions to the two following characteristic equations

$$\alpha_1^2 + K_1\alpha_1 + K_2 = 0 \quad (3.13) \qquad \alpha_2^2 - K_1\alpha_2 + K_2 = 0 \quad (3.14)$$

where

$$\alpha_{11} = -\alpha_{22} = -\frac{K_1}{2} + \sqrt{\left(\frac{K_1}{2}\right)^2 - K_2} \quad (3.15) \qquad \alpha_{12} = -\alpha_{21} = -\frac{K_1}{2} - \sqrt{\left(\frac{K_1}{2}\right)^2 - K_2} \quad (3.16).$$

Eq. (3.13) refers to the choice of the positive sign whereas Eq. (3.14) refers to the choice of the negative sign.

If we now let

$$\gamma = \frac{K_1}{2} \quad (3.17) \qquad \text{and} \qquad \delta = \sqrt{K_2 - \left(\frac{K_1}{2}\right)^2} \quad (3.18)$$

we have $\alpha_{11} = -\alpha_{22} = -\gamma + j\delta \qquad \alpha_{12} = -\alpha_{12} = -\gamma - j\delta$ (3.19)

$$\alpha_{11}^2 = \alpha_{22}^2 = \gamma^2 - \delta^2 - 2j\gamma\delta \qquad \alpha_{12}^2 = \alpha_{21}^2 = \gamma^2 - \delta^2 + 2j\gamma\delta \tag{3.20}$$

and it is easy to recognize that the system is stable when $\gamma^2 - \delta^2 \le 0$, that is when

$$K_2 - 2 \cdot \left(\frac{K_1}{2}\right)^2 \ge 0 \tag{3.21}$$

Let us now start from the consideration of the solution in *steady state conditions* attained as the solution at permanent regime.

BY-PRODUCTS EMERGY IN STEADY STATE CONDITIONS

Let us assume a *constant* input

$$Em[u(t)] = Em[u(t_0)] \tag{3.22}$$

and

$$K_2 - \left(\frac{K_1}{2}\right)^2 > 0 \tag{3.23}$$

Under such asymptotically stable conditions the general dynamic solution is given by

$$\tilde{E}\,m(y(t)) = \begin{pmatrix} \tilde{E}\,m(y_1(t)) \\ \tilde{E}\,m(y_2(t)) \end{pmatrix} = \begin{pmatrix} c_{11} \\ c_{22} \end{pmatrix} \cdot e^{\begin{pmatrix} \alpha_{11} \\ \alpha_{22} \end{pmatrix}^2 t} + \begin{pmatrix} c_{12} \\ c_{21} \end{pmatrix} \cdot e^{\begin{pmatrix} \alpha_{12} \\ \alpha_{21} \end{pmatrix}^2 t} + Em_p(t) \tag{3.24}$$

where $Em_p(t)$ is a particular integral of non-homogeneous Eq. (3.9) given by

$$Em_p(t) = \frac{K_3}{K_2} \cdot Em[u(t_0)] = Em(u_0) \tag{3.25}$$

Since the system is asymptotically stable, after its pertinent transient, it achieves its permanent regime conditions described by the solution

$$\tilde{E}\,m(y(t)) = \begin{pmatrix} \tilde{E}\,m(y_1(t)) \\ \tilde{E}\,m(y_2(t)) \end{pmatrix} = \begin{pmatrix} Em(u_0) \\ Em(u_0) \end{pmatrix} \tag{3.26}$$

which shows the validity of Odum's Rule (as originally formulated) in *steady state conditions*, when understood as being reached after a dynamic transient.

For the sake of completeness, we will also analyze the validity of the Rule both under *stationary* and *variable conditions*.

BY-PRODUCTS EMERGY IN STATIONARY CONDITIONS

For the sake of simplicity we may continue to suppose a *constant* input. Let us now assume that

$$K_2 - 2 \cdot \left(\frac{K_1}{2}\right)^2 = 0 \tag{3.27}$$

(see condition (3.21)). Consequently the system stabilizes in a *stationary* regime which depends on the assumed initial conditions. Eq. (3.9), although of the first order, has *two* initial conditions concerning both the function and its derivative of order ½ respectively, that is

$$\begin{pmatrix} Em(y_1(0)) \\ Em(y_2(0)) \end{pmatrix} = \begin{pmatrix} Em_{10} \\ Em_{20} \end{pmatrix} \quad (3.28) \quad \text{and} \quad \begin{pmatrix} Em^{(1/2)}(y_1(0)) \\ Em^{(1/2)}(y_2(0)) \end{pmatrix} = \begin{pmatrix} +Em_0^{(1/2)} \\ -Em_0^{(1/2)} \end{pmatrix} \quad (3.29).$$

Eq. (3.29) presents opposite values only because this is one of the possibilities offered by the definition of fractional derivative of order ½. Eq. (3.28), on the other hand, does not give any information about the two (theoretically distinct) values.

In this respect the interpretation of the *physical meaning* of a fractional derivative of order ½ plays a fundamental role. In fact, on the basis of its specific definition, the fractional derivative of order ½ requires that (Giannantoni, 2001b)

$$Em^{(1/2)}(0) \circ Em^{(1/2)}(0) = Em(0) \circ Em^{(1)}(0) = Em^{(1)}(0) \circ Em(0) \tag{3.30}$$ [2].

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[2] The symbol "o" indicates the "circle product" (Giannantoni, 2002a, p. 178) which represents a generalization of the concept of "product" already known in Mathematical Analysis.

This shows that the initial condition pertaining to the derivative of order ½ corresponds to an *equivalent condition* for the derivative of order 0 and order 1, respectively, those derivatives whose *integer* orders define the minimum interval that includes the *fractional* order ½.
Condition (3.30), when explicitly expressed by means of Eq. (3.24), leads to the conditions

$$c_{11} = c_{22} \qquad \text{and} \qquad c_{12} = c_{21} \tag{3.31}$$

which, on the basis of Eq. (3.24), imply that the two values defining the initial condition (3.28) must be equal, that is:

$$\begin{pmatrix} Em(y_1(0)) \\ Em(y_2(0)) \end{pmatrix} = \begin{pmatrix} Em_0 \\ Em_0 \end{pmatrix} \tag{3.32}$$

Such a condition exactly corresponds to the Rule concerning by-products, when these are considered at the initial time $t = 0$. However the basic difference is that now the rule is obtained from the same concept of a "binary" system or a co-productive binary process.
On the basis of the initial conditions (3.32) and (3.29), the general solution to Eq. (3.9) is then given by

$$\tilde{E}\,m(y(t)) = \begin{pmatrix} Em_0 - Em(u_0) \\ Em_0 - Em(u_0) \end{pmatrix} \cdot \cos\begin{pmatrix} 2\gamma\delta \\ 2\gamma\delta \end{pmatrix} t + \begin{pmatrix} \dfrac{Em_0^{(1/2)}}{+\sqrt{2\gamma\delta}} \\ \dfrac{Em_0^{(1/2)}}{+\sqrt{2\gamma\delta}} \end{pmatrix} \cdot \sin\begin{pmatrix} 2\gamma\delta \\ 2\gamma\delta \end{pmatrix} t + \begin{pmatrix} Em(u_0) \\ Em(u_0) \end{pmatrix} \tag{3.33}$$

which represents a *binary function*, the two components of which are always *equal to each other*. Each component is a function made up of two distinct sinusoidal modes, which oscillate with a difference of phase of $\pm\pi/2$ (according to the sign of the difference $Em_0 - Em(u_0)$) and, as a global result, they give rise to identical functions with the same time mean value $Em(u_0)$.
However we could alternatively choose the initial conditions in a different way, for instance as follows

$$\begin{pmatrix} Em(y_1(0)) \\ Em(y_2(0)) \end{pmatrix} = \begin{pmatrix} Em_0 \\ Em_0 \end{pmatrix} \tag{3.34}$$

and

$$\begin{pmatrix} Em^{(1/2)}(y_1(0)) \\ Em^{(1/2)}(y_2(0)) \end{pmatrix} = \begin{pmatrix} Em_0^{(1/2)} \\ Em_0^{(1/2)} \end{pmatrix} \tag{3.35}$$

that is with the derivative of order ½ having equal components. In fact Eq. (3.30) is a sort of "quadratic" form. For the same reason it continues to require equal initial values in Eq. (3.34).
The corresponding solution is now given by

$$\tilde{E}\,m(y(t)) = \begin{pmatrix} Em_0 - Em(u_0) \\ Em_0 - Em(u_0) \end{pmatrix} \cdot \cos\begin{pmatrix} 2\gamma\delta \\ 2\gamma\delta \end{pmatrix} t + \begin{pmatrix} \dfrac{Em_0^{(1/2)}}{+\sqrt{2\gamma\delta}} \\ \dfrac{Em_0^{(1/2)}}{-\sqrt{2\gamma\delta}} \end{pmatrix} \cdot \sin\begin{pmatrix} 2\gamma\delta \\ 2\gamma\delta \end{pmatrix} t + \begin{pmatrix} Em(u_0) \\ Em(u_0) \end{pmatrix} \tag{3.36}$$

which still represents a *binary function*. Each component is made up of two distinct sinusoidal modes: the former are identical for both functions, whereas the latter are always in opposition of phase to each other. In addition, the first mode of each component oscillates with a difference of phase of $\pm\pi/2$ with respect to the second one (according to the sign of the difference $Em_0 - Em(u_0)$) and, as a global result, they give rise to two oscillating functions with the same time mean value $Em(u_0)$.
In the most general case, in which the conditions on the derivative of order ½ are completely different from each other, we can always reduce it to a combination to the two previous ones by recognizing that it is always possible to decompose such a condition as follows

$$\begin{pmatrix} Em^{(1/2)}(y_1(0)) \\ Em^{(1/2)}(y_2(0)) \end{pmatrix} = \begin{pmatrix} Em_{10}^{(1/2)} \\ Em_{20}^{(1/2)} \end{pmatrix} = \begin{pmatrix} Em_a^{(1/2)} \\ Em_a^{(1/2)} \end{pmatrix} + \begin{pmatrix} +Em_b^{(1/2)} \\ -Em_b^{(1/2)} \end{pmatrix} \quad (3.37)$$

where

$Em_a^{(1/2)} = (Em_{10}^{(1/2)} + Em_{20}^{(1/2)})/2$ (3.38) and $Em_b^{(1/2)} = (Em_{10}^{(1/2)} - Em_{20}^{(1/2)})/2$ (3.39).

We can thus conclude that, in stationary (stable) conditions, the two components of output Emergy are always equal to each other, either instantaneously or as a time mean value over a period. Their values instantaneously differ from input Emergy because of the internal dynamics of the process (in particular because of the accumulation term).[3]

BY-PRODUCTS" EMERGY IN VARIABLE CONDITIONS

The case of variable conditions can be better dealt with in terms of *prior operators* such as the *generator* $\tilde{D}$. In fact the explicit solution to Eq. (3.9) can be formally written as (Giannantoni 2001d, 2002a)

$$\tilde{E}\,m(t) = \varphi(t,\tilde{D})^{-1} \cdot [\sum_{k=0}^{1} Em_0^{\left(\frac{k}{2}\right)} \tilde{D}^{\frac{1-k}{2}} \tilde{\delta}(t) + Em(u(t))] \quad (3.40)$$

where

$$\varphi(t,\tilde{D}) = (\tilde{D}^{\frac{1}{2}})^2 + K_1 \cdot D^{\frac{1}{2}} + K_2 \quad (3.41)$$

with the initial conditions already included into Eq. (3.40). The corresponding explicit solution can be then expressed as follows

$$\tilde{E}\,m(t) = \tilde{\int_0^1} \tilde{k}(t,\tau) \cdot \sum_{k=0}^{1} Em_0^{\left(\frac{k}{2}\right)} \tilde{D}^{\frac{1-k}{2}} \tilde{\delta}(\tau) \cdot \tilde{d}\tau + \tilde{\int_0^1} \tilde{k}(t,\tau) \cdot Em(u(\tau)) \cdot \tilde{d}\tau \quad (3.42)$$

where $\tilde{k}(t,\tau)$ is the "incipient" *solving kernel* of Eq. (3.9), which has always an *explicit form* based on the particular integrals on the second side of Eq. (3.12).

The first term on the second side of Eq. (3.42) represents the transient response in the *state* (sometimes termed as "free evolution in the *state*") whereas the second term represents the so-called response at permanent regime.

If the system is decisively stable (Eq. (3.21) different from zero), output Emergy, after its pertinent transient, reduces to

$$\tilde{E}\,m(t) = \tilde{\int_0^1} \tilde{k}(t,\tau) \cdot Em(u(\tau)) \cdot \tilde{d}\tau \quad (3.43),$$

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[3] However a deeper analysis of the process would easily show that such a discrepancy from what the second rule states only depends on our erroneous assumption of the model pertaining to the "tank of information" (or accumulation term). In fact it has been modeled as if it were a *mechanical* reservoir. In living systems, on the contrary, the tanks of information are of a *different nature*. They must be thus modeled (at least) as "binary" accumulation terms. This means that the term $\tilde{d}\,Em(t)/\tilde{d}\,t$ has to be replaced by $((\tilde{d}/\tilde{d}\,t)^{1/2})^2\,Em(t)$. In such a case the second rule continues to be valid, *at any time*.

which represents either an oscillating (stable) trend around a mean value defined by the instantaneous input $Em(u(t))$ or an asymptotic trend, which tends to a value defined by the same instantaneous input $Em(u(t))$. In addition, the presence of the prior generator $(\tilde{D})^{1/2}$ in Eq. (3.41) defines the *multiplicity* of the output Emergy which, in the asymptotic trend, assumes a structure which corresponds to Odum's co-production Rule.

BY-PRODUCTS EMERGY OF MORE COMPLEX CO-PRODUCTION PROCESSES

More complex co-productive systems are represented by *generative functions* such as

$$\varphi_m(t,\tilde{D}) = A_n \cdot (\tilde{D}^{\frac{1}{2}})^m + A_{n-1} \cdot (\tilde{D}^{\frac{1}{2}})^{m-1} + ...A_1 \cdot (\tilde{D}^{\frac{1}{2}})^1 + A_0 \qquad (3.44).$$

The right-hand side of Eq. (3.41) is replaced by a polynomial of order m in the prior *generator* $\tilde{D}^{\frac{1}{2}}$. In such a case output Emergy is *still* a binary function (generated by the basic incipient derivative of order ½), but at the same time it is characterized by m distinct different modes.
The analysis of such Systems does not present particular difficulties if it is dealt with in terms of the generator $\tilde{D}$, because (as we already know) the Incipient Fractional Calculus always allows us to express the solution in the formal way similar to Eq. (3.40), that is

$$\tilde{E}\,m(t) = \varphi_m(t,\tilde{D})^{-1} \cdot [\sum_{k=0}^{m-1} Em_0^{\left(\frac{k}{2}\right)}\, \tilde{D}^{\frac{m-1-k}{2}}\, \tilde{\delta}(t) + Em(u(t))] \qquad (3.45)$$

and in explicit terms as follows

$$\tilde{E}\,m(t) = \int_0^{\tilde{1}} \tilde{k}_m(t,\tau) \cdot \sum_{k=0}^{n-1} Em_0^{\left(\frac{k}{2}\right)}\, \tilde{D}^{\frac{n-1-k}{2}}\, \tilde{\delta}(\tau) \cdot \tilde{d}\tau + \int_0 \tilde{k}_m(t,\tau) \cdot Em(u(\tau)) \cdot \tilde{d}\tau \qquad (3.46).$$

The analysis can also be further generalized to a process which co-produces n distinct by-products. For these more complex systems we can repeat the same considerations made in the case of simple co-productive systems. In fact their behavior is fundamentally due to the *physical meaning* of all the fractional derivatives, for which conditions analogous to Eq. (3.30) hold. We can consequently conclude that:

i) Odum's co-production Rule is always valid under *steady state* conditions, when these are understood as a permanent regime reached after a transient with constant input;
ii) it is also valid, as a mean value, in the case of *stationary* conditions characterized by an oscillating trend around a constant value (still in the case of constant input)
iii) it is even valid under *variable conditions*, when the outputs are understood in terms of mean values (in the case of oscillating solutions) or as an amplification of n times as much the input Emergy, in the case of a decisively stable response to a variable input (when the associated transient is finished).

In all cases Odum's Maximum Em-Power Principle, which asserts the general tendency toward the Maximum of processed Emergy (see Giannantoni 2001c, 2002a), is always satisfied. This due to an increase of *Ordinality* of output Emergy (with respect to the input one) due to the *generative capacity* of the process, which gives rise to a multiple binary function. The crucial role of Ordinality will be presented , in detail, in the next paragraph.

OUTPUT CO-PRODUCTION TRANSFORMITY AND INCREASED ORDINALITY OF OUTPUT EMERGY

For the sake of simplicity let us consider the case of two co-products modeled by Eq. (3.33). In this case, the perfect identity of the terms which define the three added binary functions enables us to easily separate the *Ordinality* of the binary structure from its *conjugated cardinality*. We may thus start by writing

$$\tilde{E}\,m(y(t)) = \begin{pmatrix} \tilde{E}\,m(y_1(t)) \\ \tilde{E}\,m(y_2(t)) \end{pmatrix} =$$

$$= \begin{pmatrix}1\\1\end{pmatrix}(Em_0 - Em(u_0))\cdot\cos\begin{pmatrix}2\gamma\delta\\2\gamma\delta\end{pmatrix}t + \begin{pmatrix}1\\1\end{pmatrix}\frac{Em_0^{(1/2)}}{+\sqrt{2\gamma\delta}}\cdot\sin\begin{pmatrix}2\gamma\delta\\2\gamma\delta\end{pmatrix}t + \begin{pmatrix}1\\1\end{pmatrix}Em(u_0) =$$

$$= \begin{pmatrix}1\\1\end{pmatrix}\cdot\left\{(Em_0 - Em(u_0))\cdot\cos\begin{pmatrix}2\gamma\delta\\2\gamma\delta\end{pmatrix}t + \frac{Em_0^{(1/2)}}{\sqrt{2\gamma\delta}}\cdot\sin\begin{pmatrix}2\gamma\delta\\2\gamma\delta\end{pmatrix}t + Em(u_0)\right\} \quad (3.47).$$

Such an equation immediately shows that, at the initial time ($t = 0$), we have

$$\tilde{E}\,m(y(t)) = \begin{pmatrix} \tilde{E}\,m(y_1(t)) \\ \tilde{E}\,m(y_2(t)) \end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}\cdot\{Em(u_0)\} = \begin{pmatrix}1\\1\end{pmatrix}\cdot\left\{Tr_{\phi,u_0}\cdot\bar{E}\,x(u_0)\right\} \qquad (3.48)$$

where $\bar{E}\,x(u_0)$ represents the *total* Exergy spent to generate the input, whereas the term Tr_{ϕ,u_0} accounts for the associated previous generative processes.
Analogously we may thus re-write Eq. (3.47) as follows

$$\tilde{E}\,m(y(t)) = \begin{pmatrix}1\\1\end{pmatrix}\cdot Tr_{\phi,u_0}\left\{(\bar{E}x_0 - \bar{E}x(u_0))\cdot\cos\begin{pmatrix}2\gamma\delta\\2\gamma\delta\end{pmatrix}t + \frac{\bar{E}x_0^{(1/2)}}{\sqrt{2\gamma\delta}}\cdot\sin\begin{pmatrix}2\gamma\delta\\2\gamma\delta\end{pmatrix}t + \bar{E}x(u_0)\right\} =$$

$$= Tr_{\phi,y}\cdot[Tr_{ex}(t)\cdot Ex(y(t))] \qquad (3.49)$$

where

$$Tr_{\phi,y} = \begin{pmatrix}1\\1\end{pmatrix}\cdot Tr_{\phi,u_0} \qquad (3.50),$$

whereas $Ex(y(t))$ is the total *instantaneous* output Exergy and $Tr_{ex}(t)$ is the factor accounting for all the dissipations due to the genesis of the corresponding form of output Exergy.
In this way it becomes particularly clear that the subdivision of Transformity in two factors (initially introduced in Giannantoni 2001c,d)

$$Tr = Tr_{\phi}\cdot Tr_{ex} \qquad (3.51)$$

enables us to distinguish between the dimensionless scalar contribution due to losses of Exergy (Tr_{ex}) from the dimensional one accounting for the "emerging" of higher forms of Ordinality (Tr_{ϕ}).
The same procedure (here extremely simplified by the fact that the two by-products always have the same Emergy) can also be applied to the other considered cases by adopting a new type of Algebra, termed as "*Algebra by Ordinal-cardinality*", which defines the rules and procedures according to which it is possible to handle mathematical entities characterized by both *Ordinality* and its *conjugated cardinality*.

Such a generalized form of Algebra is also able to show that, even if the Exergy of the system tends to be dissipated, Emergy on the contrary, in *generative processes*, tends to increase. Such an increase is explicitly accounted for by Transformity. In fact the latter passes from a simple *algebraic value* to a *binary function*, which consequently shows the corresponding increase in Ordinality of the system. This, on the other hand, is nothing but what we anticipated in (Giannantoni 2002a, p. 97) when the Maximum Em-Power Principle was interpreted as a *tendency* Principle toward the Maximum of Ordinality.

SPLITS AS "DUAL" FUNCTIONS

"Splits" have already been dealt with in (Giannantoni, 2001b,d). They are thus simply recalled only to point out some aspects which stress, even more, the *deep difference* with respect to a co-production process. In fact, as already shown (ib.), any attempt at modeling a *co-production* process in terms of *two distinct* integer-order differential equations is destined to fail. *Even if* we adopt *incipient* derivatives in writing the pertinent vector equation

$$C \cdot \frac{\tilde{d}^2}{\tilde{d}t^2}\begin{bmatrix} Em_1(t) \\ Em_2(t) \end{bmatrix} + B \cdot \frac{\tilde{d}}{\tilde{d}t}\begin{bmatrix} Em_1(t) \\ Em_2(t) \end{bmatrix} + A \cdot \begin{bmatrix} Em_1(t) \\ Em_2(t) \end{bmatrix} = Em(u(t)) \qquad (4.1)$$

and *even if* we require that the associated initial conditions

$$\begin{bmatrix} Em_1(0) \\ Em_2(0) \end{bmatrix} = \begin{bmatrix} Em_0 \\ Em_0 \end{bmatrix} \qquad (4.2) \quad , \qquad \begin{bmatrix} Em_1'(0) \\ Em_2'(0) \end{bmatrix} = \begin{bmatrix} Em_{10}' \\ Em_{20}' \end{bmatrix} \qquad (4.3)$$

satisfy Eq. (3.30) (the first member of which is defined on the basis of the initial binary function) the general solution to Eq. (4.1) can never ever coincide with the solution to Eq. (3.9). The reason is that a *fractional* differential problem is never reducible to an *integer-order* differential problem (although thought of in terms of incipient derivatives) without losing its specific intrinsic characteristics. This is because the definition of a binary function depends *only* on the *specific type* of the *unique* differential equation to which it is solution. More precisely, it only depends *on the fractional derivative of order 1/2*. Certainly it might "degenerate" into a "dual" function, that is a function made up of two independent "monadic"[2] functions (*extrinsically* related, as in the case of a vector). However this happens *only* if we *decide beforehand* to analyze the trend of such (supposedly independent) solutions *exclusively in terms of integer-order derivatives* (that is on the basis of a particular perspective, *preliminarily* chosen, which implicitly *excludes* other possibilities).

Consequently, the behavior described by *one fractional incipient* differential equation *uniquely* characterizes (and consequently *defines*) a *generative co-production* process, whereas a *vector* differential equation (although in terms of incipient derivatives) is only able to describe a *split* process (because its solutions are only extrinsically related, and thus termed as "dual" functions). This immediately implies that the two distinct processes co-production and split *can never be confused*, because their specific definition is now based on the *intimate generative structure* of each process, which is uniquely described in differential terms: co-production is represented by *fractional basic* incipient derivatives and a split by *integer basic* incipient derivatives.

INTER-ACTION

Interaction constitutes another fundamental *generative* process. It is generally symbolized as in Fig. 5.1, which evidently represents the simplest form of interaction, even if its elementary structure can also be the basis for more complex interactions. Its output structure is generally associated with *non-linear* input or process dynamics (or both). An example of input-output non-linear dynamics can be given by Riccati's Equation which, written in terms of incipient derivatives,

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[2] That is each one can be thought of as a solution of an independent *integer-order* differential equation.

$$\frac{\tilde{d}}{\tilde{d}\,t} f(t) + Q(t) f(t) + R(t) f^2(t) = P(t) \qquad (5.1)$$

can be diagrammatically represented as in Fig. 5.2, where $G(t, \tilde{D}) = \varphi(t, \tilde{D})^{-1}$ is the *transference function* of the intrinsically linear process.

Riccati's Equation furnishes a solution in form of a "duet" (thus represented as $[\tilde{f}(t), \tilde{f}(t)]$) because it is understood as the result of a "*circle product*" (or *product* by *Ordinal-cardinality*) between two traditional functions, each one of Ordinality (1), which gives rise to a completely new "*function*", of Ordinality (2). In fact, by definition of *circle product* (indicated by "$\circ$" and defined in (Giannantoni 2002a, p. 178)), we have

$$[\tilde{f}(t), \tilde{f}(t)] = \tilde{f}(t) \circ \tilde{f}(t) = \tilde{f}(t)^{1,(1)} \circ \tilde{f}(t)^{1,(1)} = \tilde{f}(t)^{2,(2)} \qquad (5.2)$$

where the "exponents" *in round brackets* indicate the *degree of Ordinality* of the corresponding "functions".
Riccati's Equation is here explicitly recalled because Odum (1994a, p. 147) used it as an example of the simplest *self-organizing* system because it is based on the simplest self-interaction *generative* process. Now, by taking into account that its formulation in terms of incipient derivatives implicitly transforms this *non-linear* equation into an *intrinsically linear* one (although in terms of "duet" functions), we can always separate such a "duet" non-linearity from the remaining linear dynamics (see Fig. 5.2). Such a result can also be generalized to *any* interaction represented by the scheme in Fig. 5.1. Consequently any interaction can always be considered as made up of a "duet" non-linearity input, followed by a linear dynamic process (in terms of output quantity) characterized by a differential equation of order *n*, as represented in Fig. 5.3, where $G(t, \tilde{D}) = \varphi_n(t, \tilde{D})^{-1}$ is the *transfer function* of the linear process modeled in terms the *generator* $\tilde{D}$.
As is well-known, output Emergy is then the result of two contributions: the first one pertaining to the transient regime, the second one describing the permanent regime

$$\tilde{E}m[\tilde{y}(t)] = \varphi_n(t, \tilde{D})^{-1} \cdot (\sum_{k=0}^{n-1} c_k \tilde{D}^{n-1-k}) \tilde{\delta}(t) + \varphi_n(t, \tilde{D})^{-1} \cdot \tilde{E}m[\tilde{u}_1(t)] \cdot \tilde{E}m[\tilde{u}_2(t)] \qquad (5.3).$$

If the process is stable, the first term on the second side of Eq. (5.3) progressively tends to zero, so that, at permanent regime, we are left with

$$\tilde{E}m[\tilde{y}(t)] = \varphi_n(t, \tilde{D})^{-1} \cdot \tilde{E}m[\tilde{u}_1(t)] \cdot \tilde{E}m[\tilde{u}_2(t)] =$$
$$= \tilde{\int_0^1} \tilde{k}(t, \tau) \cdot \tilde{E}m[\tilde{u}_1(\tau)] \cdot \tilde{E}m[\tilde{u}_2(\tau)] \cdot \tilde{d}\tau \qquad (5.4)$$

where the last term expresses the explicit result by means of the solving kernel $\tilde{k}(t, \tau)$.
If we now assume that the system is stable as a consequence of a transfer function characterized by *all real roots*, Eq. (5.4) tends to the structure

$$\tilde{E}m[\tilde{y}(t)] = k_{\text{int}} \cdot \tilde{E}m[\tilde{u}_1(t)] \cdot \tilde{E}m[\tilde{u}_2(t)] \qquad (5.5)$$

where k_{int} is a dimensional factor corresponding to the integration of the solving kernel when $t \to \infty$. The result obtained (Eq. (5.5)) evidently coincides with the initial assumption made by Odum about the interaction process (see Fig. 5.1).
If, on the contrary, the system, although stable, has a transfer function characterized by *couples of complex roots*, the coefficient k_{int} becomes a periodic function of time. This simplified analysis of the interaction (which can be easily generalized to the various cases previously considered with reference to a co-production process) is already sufficient to show the increase in Ordinality of the output Emergy.

INTER-ACTION OUTPUT TRANSFORMITY AND INCREASED ORDINALITY OF OUTPUT EMERGY

If, in analogy to the case of a co-production process, we do not consider the Ordinality of Exergy (as usual happens for physical quantities) but only the Ordinality pertaining to Transformity, we have

$$\tilde{E}m[\tilde{y}(t)] = k_{\text{int}} \cdot \tilde{E}m[\tilde{u}_1(t)] \circ \tilde{E}m[\tilde{u}_2(t)] =$$

$$= k_{\text{int}} \cdot [\tilde{T}r_{\phi,1} \cdot \tilde{T}r_{ex.1} \cdot \tilde{E}x(\tilde{u}_1)] \circ [\tilde{T}r_{\phi,2} \cdot \tilde{T}r_{ex.2} \cdot \tilde{E}x(\tilde{u}_2)] =$$

$$= k_{\text{int}} \cdot [\tilde{T}r_{\phi,1} \circ \tilde{T}r_{\phi,2}] \cdot [(\tilde{T}r_{ex.1} \cdot \tilde{E}x(\tilde{u}_1)) \cdot (\tilde{T}r_{ex.2} \cdot \tilde{E}x(\tilde{u}_2))] \qquad (5.6).$$

If we then consider the pertinent Ordinality of the two Transformities, the circular product in Eq. (5.6), evaluated according to Eq. (5.2), gives

$$[\tilde{T}r_{\phi,1} \circ \tilde{T}r_{\phi,2}] = [(\tilde{T}r_{\phi,1})^{1,(1)} \circ (\tilde{T}r_{\phi,2})^{1,(1)}] = [(\tilde{T}r_{\phi,1} \cdot \tilde{T}r_{\phi,2})^{1,(2)}] \qquad (5.7)$$

which shows that the *Interaction Process* has an *Ordinality* of order (2), that is higher than the initial ones (supposed to be equal to (1)). Interaction is thus a process which *generates an increase* in the pertinent Ordinality of its output Emergy.
In other words, under dynamic conditions, the interaction not only presents a quantitative gain greater than 1 (due to the contribution of internal source terms), but also an increase in the pertinent Ordinality of the output, which represents an excess *Quality*, not strictly reducible to its mere phenomenological premises.
We can thus conclude that the fifth Rule stated by Odum (who did not introduce explicit Ordinality notations) correctly accounts for both the *quantitative* gain of the process and the corresponding gain in *Quality*, although the latter is expressed through a *quantitative increase* in the pertinent (scalar) Transformity.

FEED-BACK

Let us now consider another *generative* process, though of a different nature: a self-organizing system characterized by an internal feedback chain as represented in Fig. 6.1.

As is well-know, the transfer function of the whole system, written in terms of the *generator* $\tilde{D}$, is given by

$$W(t,\tilde{D}) = \frac{G(t,\tilde{D})}{1 + G(t,\tilde{D}) \cdot H(t,\tilde{D})} \qquad (6.1)$$

where $$G(t,\tilde{D}) = \varphi_n(t,\tilde{D})^{-1} \qquad (6.2)$$

represents the transfer function of a differential process (supposed of order *n*), whereas $\tilde{H}(t,\tilde{D})$ is the transfer function of the feedback. The latter can be generally thought of as given by the following structure

$$H(t,\tilde{D}) = \frac{Q_q(t,\tilde{D})}{R_r(t,\tilde{D})} \tag{6.3}$$

a ratio of two polynomials in $\tilde{D}$ of order *q* and *r* respectively, where $q \le r$. The numerator $Q_q(t,\tilde{D})$ can be reduced to a constant which can be chosen to be 1 (without any lack of generality) so that Eq. (6.3) becomes

$$H(t,\tilde{D}) = \frac{1}{R_r(t,\tilde{D})} \tag{6.4}$$

If now introduce Eqs. (6.2) and (6.4) into Eq. (6.1), we get

$$W(t,\tilde{D}) = \frac{R_r(t,\tilde{D})}{\varphi_n(t,\tilde{D}) \cdot R_r(t,\tilde{D}) + 1} \tag{6.5}$$

which allows us to illustrates the principal effects of the feedback. These can be synthesized as follows:

i) a *negligible* influence *in module* on the controlled process, because

$$\left|H(\tilde{D})\right| << 1 \quad (6.6) \qquad \text{and, consequently,} \qquad \left|W(\tilde{D})\right| \cong \left|G(\tilde{D})\right| \quad (6.7)$$

ii) an *extreme relevance* in terms of both *stability* and *Ordinality* of the output Emergy.

The stabilization effect is due to the "translation" of the *n* zeros of the function $\varphi_n(t,\tilde{D})$ which characterizes the dynamics of the process. The polynomial in $\tilde{D}$ at the denominator of Eq. (6.5), as a consequence of the term $R_r(t,\tilde{D})$ due to the feedback, has now *n* + *r* complex roots with their real part less than zero, which yields stability.

The effect on output Ordinality is analyzed in more detail in the next paragraph.

FEED-BACK OUTPUT TRANSFORMITY AND INCREASED ORDINALITY OF OUTPUT EMERGY

In the simple case of a *constant input* $Em(u_0)$, the response at *permanent regime*, under the hypothesis of *asymptotically stable conditions*, is given by

$$\tilde{E}m[y(t)] = W(\tilde{D}) \cdot \tilde{E}m(u_0) = \frac{R_r(\tilde{D})}{\varphi_n(\tilde{D}) \cdot R_r(\tilde{D}) + 1} \cdot \tilde{E}m(u_0) = W_0^{1,(r),[n+r]} \cdot \tilde{E}m(u_0) \tag{6.8}$$

where

$$\left|W_0\right| \cong 1 \tag{6.9}$$

Eq. (6.8) shows an increase in Ordinality of output Emergy due to both the incipient integration of order $n + r$ (*compositive* Ordinality) and the incipient differentiation of order r (multiple "duet" Ordinality).
Such a higher level of Ordinality is faithfully taken into account by the *generative* Transformity. In fact, under the same conditions, Eq. (6.8) can be written as follows

$$\tilde{E}m[y(t)] = Tr_{\phi,y} \cdot [Tr_{ex,y} \cdot \tilde{E}x_y] = W_0^{1,(r),[n+r]} \cdot Tr_{\phi,u_0} \cdot [Tr_{ex,u_0} \cdot \tilde{E}x_{u_0}] \qquad (6.10)$$

and consequently

$$Tr_{\phi,y} = W_0^{1,(r),[n+r]} \cdot Tr_{\phi,u_0} \qquad (6.11),$$

which shows that:
- output Transformity is *much richer* than the Input one in *terms of Ordinality*
- it accounts for $n + r$ basic harmonics, which are not only *harmoniously composed* among them (by integration), but are also *consonant* with all their r pertinent *genetic harmonics* (generated by the r different orders of derivation).

Similar results can be easily achieved when the analysis is generalized to *stationary* and *variable conditions* (in analogy to the cases already analyzed with reference to a co-production process).

CONCLUSIONS

The previous analysis allows us to draw the following *main* conclusions:

i) Emergy Algebra has *solid Differential Bases*, not only in *steady state conditions* (as originally conceived) but in *stationary* and *variable* conditions too;
ii) Odum's Rules are substantially correct even if Transformity, *for practical reasons*, is considered as being a simple *scalar*;
iii) *Generative Processes* (such as *co-production*, *inter-action* and *feed-back*) present an *excess Quality* in their output Emergy, which is "reflected" by the pertinent levels of Ordinality of their output Transformities;
iv) This also confirms the advantages of the subdivision of Transformity into *two distinct factors*

$$Tr = Tr_{\phi} \cdot Tr_{ex} \qquad (7.1)$$

where Tr_{ϕ} (*generative* Transformity) accounts for "emerging" forms of higher *Ordinality*, whereas Tr_{ex} (*dissipative* Tansformity) accounts for losses of Exergy;

v) in particular, the Ordinality of Tr_{ϕ} is the one which accounts for the *progressive increase in Quality*, as stated by the Maximum Em-Power Principle.

From a more general point of view, however, we can draw some additional conclusions.
The three processes previously analyzed represent the basic modalities according to which the *emerging of an excess of Emergy* takes place. This constitutes the most important of Odum's discovery in his lifetime work (see Giannantoni, 2003b). This discovery acquired its most significant expression in the formulation of the Maximum Em-Power Principle, a revised and updated version of Lokta's Principle.
The completely *new perspective* introduced by such a *new concept* of Quality also suggested the development of an appropriate mathematical language, explicitly finalized by translating Odum's ideas into a generally recognized formal language.
Initially applied to living systems, such a language has shown its validity for *non-living* systems too (see, for instance, Mercury's Precessions (Giannantoni, 2001d).
Consequently, the rules of Emergy Algebra can also be considered as being the basis for the development of a new mathematics. Such a *Mathematics for Generative Processes* represents a *radical passage* from a description of processes based on *velocity* and *acceleration* (expressed by the traditional derivatives of the first and second order), to a *new* description of the same based on their

generating capacity and associated *generation supra-abundance*, adequately represented by the *incipient* derivatives of the first and second order, respectively (Giannantoni, 2003a).

REFERENCES

Brown M. T., 1993. *Workshop on Emergy Analysis*. Siena, September 20-25.

Brown M. T. and Herendeen R. A., 1996. *Embodied Energy Analysis and EMERGY analysis: a comparative view*. Ecological Economics 19 (1996), 219-235.

Giannantoni C., 2000a. *Toward a Mathematical Formulation of the Maximum Em-Power Principle.* Proceedings of the First Biennial Emergy Analysis Research Conference. Univ. of Florida, Gainesville (USA), p. 155-169.

Giannantoni C., 2001a. *Advanced Mathematical Tools for Energy Analysis of Complex Systems.* Proceedings of the International Workshop on "Advances in Energy Studies". Porto Venere (Italy), May 23-27, 2000. Ed. S.G., Padua.

Giannantoni C., 2001b. *The Problem of the Initial Conditions and Their Physical Meaning in Linear Differential Equations of Fractional Order.* Third Workshop on "Advanced Special Functions and Related Topics in Differential Equations" - June 24-29 – Melfi (Italy). Applied Mathematics and Computation 141 (2003) 87-102. Elsevier Science.

Giannantoni C., 2001c. *Mathematical Formulation of the Maximum Em-Power Principle.* Second Biennial International Emergy Conference. Gainesville (Florida, USA), September 20-22.

Giannantoni C., 2001d. *Mathematics for Quality. Living and Non-Living Systems*. Second Emergy Evaluation and Research Conference. Gainesville (Florida, USA), September 20-22, 2001.

Giannantoni C., 2002a. *The Maximum Em-Power Principle as the basis for Thermodynamics of Quality*. Ed. S.G.E., Padova, ISBN 88-86281-76-5.

Giannantoni C., 2003a. *La Matematica dei Processi Generativi*. Convegno sul Calcolo Matematico Precolombiano. Internazionale. IILA Institute, Rome, October 21, 2003 (in the process of publication).

Giannantoni C., 2003b. *A harmonious dissonance.* Ecological Modelling, December 2003 (in the process of publication).

Kolmogorov A. N. and Fomin S. V., 1980. *Elements of the Theory of Functions and Functional Analysis*. Ed. MIR, Moscow.

Kransov M. L., 1983. *Ordinary Differential Equations*. Ed. MIR, Moscow.

Lazzarin R. and Macor A., 1989. *Introduzione all'Analisi Exergetica*. Ed. CLEUP, Padova.

Lotka A. J., 1922a. *Contribution to the Energetics of Evolution*. Proceedings of the National Academy of Sciences, 8 (1922), 147-150.

Lotka A. J., 1922b. *Natural Selection as a Physical Principle*. Proceedings of the National Academy of Sciences, 8 (1922), 151-155.

Lotka A. J., 1945. *The Law of Evolution as a Maximal Principle*. Human Biology, a record of research. Vol. 17, n. 3, September.

Odum H. T. and Pinkerton R. C., 1955. *Time Speed Regulator: the Optimum Efficiency for Maximum Power Output in Physical and Biological Systems.* American Scientist, 43 (1955), 331-343.

Odum H. T., 1983. *Maximum Power and Efficiency: a Rebuttal*. Ecological Modelling, 20 (1983), 71-82.

Odum H. T., 1988. *Self-Organization, Transformity and Information*. Science, v. 242, pp. 1132-1139, November 25.

Odum H. T., 1994a. *Ecological and General Systems. An Introduction to Systems Ecology*. Re. Edition. University Press Colorado.

Odum H. T., 1994b. *Environmental Accounting.* Environ. Engineering Sciences. University of Florida.

Odum H. T., 1994c. *Ecological Engineering and Self-Organization*. Ecological Engineering. An Introduction to Ecotechnology. Edited by Mitsch W. and Jorgensen S.. J. Wiley & Sons, Inc..

Odum H. T., 1994d. *Self Organization and Maximum Power.* Environ. Engineering Sciences. University of Florida.

Odum H. T., 1995a. *Energy Systems and the Unification of Science*. From Maximum Power. The Ideas and Applications of H. T. Odum. C. A. S. Hall, Editor. University Press Colorado.
Szargut J., Morris D. R. and Steward F. R., 1988. *Exergy Analysis of Thermal, Chemical and Metallurgical Processes*. Hemisphere Publ. Corp., USA.